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# Operational Quantities Characterizing Semi–Fredholm Operators

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A operational quantity is a procedure which determine, for every pair $X$, $Y$ of infinite dimensional Banach spaces, a map from $L(X,Y)$ (the class of all continuous linear operators from $X$ into $Y$) into the non–negative numbers. Several authors have considered operational quantities in order to obtain characterizations and perturbation results for the classes of operators of the Fredholm theory. For example, Schechter introduced in [10] operational quantities derived from the norm in the following way:

Recall that $T \in L(X,Y)$ is said to be upper semiFredholm if its range is closed and its kernel is finite dimensional, and it is said to be strictly singular if no restriction of $T$ to $M \in S(X)$ is an isomorphism, where $S(X)$ is the class of all infinite dimensional (closed) subspaces of $X$. We denote by

$$n(T) := \|T\|$$

the norm of $T \in L(X,Y)$. Schechter [10] (with a different notation) defined

$$in(T) := \inf\{n(TJ_M) : M \in S(X)\},$$

$$sin(T) := \sup\{in(TJ_M) : M \in S(X)\},$$

where $J_M$ stands for the canonical inclusion of $M$ into $X$, and proved that

$$T \text{ is upper semi–Fredholm if and only if } in(T) > 0,$$

$$T \text{ is strictly singular if and only if } sin(T) = 0,$$

and for $K, T \in L(X,Y)$,

$$sin(K) < in(T) \quad \Rightarrow \quad T + K \text{ is upper semi–Fredholm.}$$

In particular, last result unifies and improves previous results about the stability of upper semi–Fredholm operators under perturbation by small–norm and strictly singular operators (see [3]).

Given two operational quantities $a$ and $b$ we will write $a \leqslant \alpha b$, for $\alpha > 0$, if for any Banach spaces $X$, $Y$ and $T \in L(X, Y)$ we have $a(T) \leqslant \alpha b(T)$. We will say that $a$ and $b$ are comparable if $\alpha a \leqslant b$ or $\alpha b \leqslant a$ for some $\alpha > 0$; and we will say that they are equivalent if $\alpha a \leqslant b \leqslant \beta a$ for some $\beta > \alpha > 0$.

In this paper we will show that all operational quantities which have appeared until now in the literature characterizing upper semi–Fredholm operators can be divided into three classes, in such a way that two quantities are equivalent if they belong to the same class, and are comparable but not equivalent if they belong to different classes. An analogous classification will be done for operational quantities characterizing lower semi–Fredholm operators.

## I. Operational Quantities Derived from the Injection Modulus

We will consider the following families of subspaces of $X$:

$$S(X) := \{M \subset X : M \text{ is an infinite dimensional subspace of } X\},$$

$$S^*(X) := \{M \subset X : M \text{ is an finite codimensional subspace of } X\}.$$

Let $T \in L(X, Y)$. The injection modulus of $T$ is defined by

$$j(T) := \inf\{\|Tx\| : x \in X,\ \|x\| = 1\}.$$

Using the injection modulus, we define

$$s^*j(T) := \sup\{j(TJ_M) : M \in S^*(X)\} \quad [10],$$

$$sj(T) := \sup\{j(TJ_M) : M \in S(X)\} \quad [10],$$

$$isj(T) := \inf\{sj(TJ_M) : M \in S(X)\} \quad [7], [4].$$

We have [10], [7], [4]

$$T \text{ is upper semi–Fredholm } \Leftrightarrow\ s^*j(T) > 0\ \Leftrightarrow\ isj(T) > 0.$$

Next we state

THEOREM 1. *We have $s^*j \leqslant isj$; however $s^*j$ and $isj$ are not equivalent.*

Förster and Liebetrau [2] defined:

$$j_{Co}(T) := \sup\{\epsilon \geqslant 0 : \exists Z, \exists K \in Co(X, Z), \forall x \in X,\ \epsilon\|x\| \leqslant \|Tx\| + \|Kx\|\},$$

where $Z$ is a Banach space and $Co(X,Z)$ is the class of all compact operators from $X$ into $Z$. We obtain

$$s^*j(T) = j_{Co}(T) = \sup\{\epsilon \geqslant 0 : \exists K \in Co(X,Y), \forall x \in X, \ \epsilon\|x\| \leqslant \|Tx\| + \|Kx\|\}.$$

The Hausdorff or ball measure of noncompactness of a bounded subset $D$ of $X$ is defined by

$$h(D) := \inf\{\epsilon > 0 : D \subset F + \epsilon B_X \text{ for some finite subset } F \subset X\},$$

where $B_X$ is the closed unit ball of $X$. From the Hausdorff measure of noncompactness, several operational quantities have been derived.

$$h_b(T) := \inf\{h(TD) : D \subset X \text{ bounded, } h(D) = 1\} \quad [1], [6], [12], [14],$$

$$h_{cb}(T) := \inf\{h(TD) : D \subset X \text{ bounded contable, } h(D) = 1\} \quad [1], [14].$$

These quantities are equivalent to $s^*j$, hence they characterize the upper semi–Fredholm operators [1; Prop. 4, Prop. 5].

## II. Operational Quantities Derived from the Norm

In the comparation of operational quantities, the notion of distortable Banach space, as given by Schlumprecht [11], is useful. Given a number $\lambda > 1$, we will say that the infinite dimensional Banach space $(X, \|.\|)$ is $\lambda$–distortable if there exists an equivalent norm $|.|$ on $X$ such that for each subspace $M \in S(X)$ we have

$$\sup\left\{\frac{|x|}{|y|} \ : \ x, y \in M, \ \|x\| = \|y\| = 1\right\} \geqslant \lambda.$$

We will say that $X$ is arbitrarily distortable if it is $\lambda$–distortable for any $\lambda > 1$. A $\lambda$–distortion of $X$ will be an isomorphism $A$ of $X$ onto a Banach space $Z$ such that for every $M \in S(X)$ we have $\lambda j(AJ_M) \leqslant n(AJ_M)$.

James [5] proved that $c_0$ and $\ell_1$ are $\lambda$–distortable for no $\lambda > 1$, and the long–standing open question if the spaces $\ell_p$ $(1 < p < \infty)$ are distortable has been recently solved by Odell and Schlumprecht [8], showing that they are arbitrarily distortable.

THEOREM 2. *For a Banach space $X$ and $\lambda > 1$, the following assertions are equivalent:*

(*a*) *The space $X$ is $\lambda$–distortable.*

(*b*) *There exists a $\lambda$–distortion of $X$.*

(c) *There exists an isomorphism $A$ from $X$ onto a Banach space $Z$ such that for every $M \in S(X)$ we have $\lambda isj(AJ_M) \leqslant in(AJ_M)$.*

COROLLARY 3. *We have $isj \leqslant in$; however $isj$ and $in$ are not equivalent.*

Using the Hausdorff measure of noncompactness we define

$$ih(T) := \inf\{h(TB_M) : M \in S(X)\} \quad [9], [12].$$

The operational quantities $in$ and $ih$ are equivalent,

$$ih(T) \leqslant in(T) \leqslant 2ih(T) ,$$

hence $ih$ characterizes upper semi-Fredholm operators.

In the following diagram we summarize the relations between the operational quantities which characterize upper semi-Fredholm operators. The symbol $\updownarrow$ means "equivalent", and the symbol $\longrightarrow$ means "$\leqslant$, but not equivalent":

$$\begin{array}{ccccc} s^*j & \longrightarrow & isj & \longrightarrow & in \\ \updownarrow & & & & \updownarrow \\ h_b & & & & ih \\ \updownarrow & & & & \\ h_{cb} & & & & \end{array}$$

*Observation* 4. (a) The class $SF_+(X, Y)$ of all upper semi-Fredholm operators from $X$ into $Y$ is an open subset of $L(X, Y)$. Consequently $T \in L(X, Y)$ is upper semi-Fredholm if and only if its distance to the boundary $\partial SF_+(X, Y)$ is strictly positive. Moreover, $in(T)$ is smaller or equal than the distance of $T$ to $\partial SF_+(X, Y)$ [1]. Then the question arises whether both coincide, or are equivalent.

(b) It is well-known that

$$\lim(a(T^n)^{1/n}) = \inf\{|\lambda| : \lambda I - T \notin SF_+\} , \qquad (1)$$

for $a = in$, $s^*j$ [14], [12] and $T \in L(X,X)$, hence (1) holds for $a = isj$, $ih$, $h_b$, $h_{cb}$. Consequently the operational quantities characterizing the upper semi-Fredholm operators have the same asymptotic behaviour. We observe that the asymptotic behaviour of the distance to $\partial SF_+$ is given also by (1) [15].

## III. LOWER SEMI-FREDHOLM OPERATORS

Recall that $T \in L(X, Y)$ is said to be lower semi-Fredholm if its range is finite codimensional (hence closed). In this section we classify operational quantities characterizing lower semi–Fredholm operators, in a similar way as we have done before for upper semi–Fredholm operators.

We will consider the following families of subspaces of $Y$:

$$Q(Y) := \{U \subset Y : Y/U \text{ is infinite dimensional}\} ,$$

$$Q_*(Y) := \{U \subset Y : U \text{ is a finite dimensional subspace of } Y\} .$$

We denote by $Q_U$ the quotient map of $Y$ onto $Y/U$. From the surjection modulus

$$q(T) := \sup\{\epsilon > 0 : \epsilon B_Y \subset TB_X\}$$

of $T \in L(X, Y)$, the following quantities are derived:

$$s_*q'(T) := \sup\{q(Q_U T) : U \in Q_*(Y)\} \quad [14],$$

$$sq'(T) := \sup\{q(Q_U T) : U \in Q(Y)\} \quad [14],$$

$$isq'(T) := \inf\{sq'(Q_U T) : U \in Q(Y)\} \quad [7], [14] .$$

We have [14], [7], [4]:

$$T \text{ is lower semi–Fredholm} \quad \Leftrightarrow \quad s^*q'(T) > 0 \quad \Leftrightarrow \quad isq'(T) > 0 .$$

THEOREM 5. *We have* $s^*q' \leqslant isq'$*; however,* $s^*q'$ *and* $isq'$ *are not equivalent.*

Förster and Liebetrau [2] defined, for $T \in L(X, Y)$,

$$q_{Co}(T) := \sup\{\epsilon \geqslant 0 : \exists Z, \exists K \in Co(Z, Y), \epsilon B_Y \subset TB_X + KB_Z\} .$$

We obtain

$$s_*q'(T) = q_{Co}(T) = \sup\{\epsilon \geqslant 0 : \exists K \in Co(X, Y), \epsilon B_Y \subset TB_X + KB_X\} .$$

For $T \in L(X, Y)$, Weis [13] defined

$$in'(T) := \inf\{n(Q_U T) : U \in Q(Y)\} ,$$

and proved that $T$ is lower semi–Fredholm if and only if $in'(T) > 0$.

THEOREM 6. *We have* $isq' \leqslant in'$ *; however,* $isq'$ *and* $in'$ *are not equivalent.*

Tylli [12] proved that the operational quantity $ih'$ given by

$$ih'(T) := \inf\{h(Q_U TB_X) : U \in Q(Y)\} ,$$

coincides with $in'$.

*Observation* 7. (a) The class $SF_-(X,Y)$ of all lower semi–Fredholm operators from $X$ into $Y$ is an open subset of $L(X,Y)$, and $in'(T)$ is smaller or equal than the distance of $T$ to the boundary $\partial SF_-(X,Y)$. Then the question arises whether $in'$ and the distance to $\partial SF_-(X,Y)$ coincide, or are equivalent.

(b) For $a = in'$, $s_*q'$ and $T \in L(X,X)$, we have [14]

$$\lim(a(T^n)^{1/n}) = \inf\{|\lambda| : \lambda I - T \notin SF_-\} , \qquad (2)$$

Hence (2) also holds for $a = isq'$. Note that (2) holds also for the distance to $\partial SF_-$ [15].

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