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# On Uniformly Dense and m–Dense Subsets of $C(X)$

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For a completely regular space $X$, $C(X)$ $(C^*(X))$ denotes the algebra of all real – valued continuous (continuous and bounded) functions over $X$. We consider two topologies on $C(X)$, namely the uniform topology ($u$ – topology) and the $m$ – topology.

The $u$ – topology is defined on $C(X)$ by taking as a base for the neighborhood system at $f$ all sets of the form

$$\{g \in C(X) : |f-g| < \epsilon\}$$

where $\epsilon$ is a positive real number. $C(X)$ endowed with this topology is a topological $l$ – group (lattice + group).

The $m$ – topology is defined on $C(X)$ by taking as a base for the neighborhood system at $f$ all sets of the form

$$\{g \in C(X) : |f-g| < u\}$$

where $u$ is a positive unit of $C(X)$. Now $C(X)$ is a topological ring.

If $X$ is not pseudocompact (i.e., $C(X) \neq C^*(X)$), $C(X)$ is not a topological vector space with any of two topologies, but they are many useful in the study of certain problems in Functional Analysis, in Topology,... For instance, in the algebraic and topological characterization of $C(X)$ (Anderson [1], Hewitt [6], Hager [5], Garrido [3],...) or in the calculus of the spectrum of subalgebras of $C(X)$ (Llavona [8], Jaramillo [7], Lindstrom [2],...).

Obviously the $m$ – topolgy is finer than the $u$ – topology and the two coincide if and only if $X$ is pseudocompact (Gillman – Jerison [4]). Although these topologies are different, when $X$ is not pseudocompact, many families in $C(X)$ that are $u$ – dense are $m$ – dense too. For instance, every subalgebra of $C(X)$ closed under (bounded) inversion is $u$ – dense if and only if it is $m$ – dense (Kurzweil [9]).

The remaining question is if the Kurzweil result is valid for another algebraic structures, like linear subspaces, subrings,... In this paper we prove that the answer is negative for linear subspaces of $C(X)$.

THEOREM. *The following conditions are equivalent:*

*a) $X$ is pseudocompact.*

*b) Every $u$–dense linear subspace of $C(X)$ is $m$–dense.*

*Proof.* *a)* $\Longrightarrow$ *b).* It is obvious because in this case the two topologies are the same. *b)* $\Longrightarrow$ *a).* If $X$ is not pseudocompact then $\beta X \neq \nu X$ ($\beta X$ is the Stone – Cech compactification of $X$ and $\nu X$ is the Hewitt – Nachbin realcompactification of $X$). Let $p \in \beta X - \nu X$. Since $\nu X$ is realcompact, there exists $h^{\beta} \in C(\beta X)$ with $0 \leqslant h^{\beta} \leqslant 1$, $h^{\beta}(p) = 0$ and $h^{\beta}(p) > 0$ if $x \in \nu X$. Let $h = h^{\beta}{}_{|X}$ and $O_p = \{f \in C^*(X) : Z(f^{\beta})$ is a neighborhood of $p\}$ (if $f \in C^*(X)$ then $f^{\beta}$ is its extension to $\beta X$ and $Z(f^{\beta}) = \{q \in \beta X : f^{\beta}(q) = 0\}$).

It is easy to prove that:

1. – $O_p$ is a linear subspace of $C(X)$ and $h \notin O_p$.

2. – If $0 < g(x) \leqslant h(x)$ for every $x \in X$, then $g \in \overline{O_p} - O_p$ ( $\overline{O_p}$ is the uniform closure of $O_p$).

Let $\mathfrak{L} = \{$linear subspaces of $C(X)$ which contain $O_p$ but not contain any $g$ with $0 < g \leqslant h\}$. $\mathfrak{L}$ is not empty because $O_p$ belongs to $\mathfrak{L}$. Applying the Zorn lemma to the family $\mathfrak{L}$ we find a maximal member $\mathfrak{F}$.

$\mathfrak{F}$ is not $C(X)$ and it is $m$ – closed because if $\mathfrak{F}^m$ is the $m$ – closure of $\mathfrak{F}$ then $\mathfrak{F}^m$ belongs to $\mathfrak{L}$ anf then $\mathfrak{F} = \mathfrak{F}^m$.

On the other hand $\mathfrak{F}$ is $u$ – dense. If $f \in C(X) - \mathfrak{F}$ there is $g \in \mathfrak{F}$ and $\lambda \in \mathbb{R}$ such that $0 < g + \lambda f \leqslant h$ and so $g + \lambda f \in \overline{O_p} \subset \overline{\mathfrak{F}}$. Since $f = \lambda^{-1}(g + \lambda f - g)$ then $f$ is in $\overline{\mathfrak{F}}$. ∎

So that for subalgebras of $C(X)$ closed under (bounded) inversion, $u$ – dense and $m$ – dense are equivalent (Kurzweil [9]) but not for linear subspaces. Actually we are trying analogous results with another algebraic structures.

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