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Absolutely (∞,p) Summing and Weakly-p-compact operators in Banach spaces

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A sequence (x_n) in a Banach space X is said to be weakly-p-summable, $1\leqslant p<+\infty$, when for each $x^*\in X^*$, $(x^*x_n)\in \ell_p$. We shall say that a sequence (x_n) is weakly-p-convergent if for some $x\in X$, (x_n-x) is weakly-p-summable.

Lemma The following statements regarding a formal series $\Sigma_n x_n$ in a Banach space are equivalent:

1. $\Sigma_n x_n$ is weakly-p-summable, $1\leqslant p<+\infty$

2. There is a C>0 such that for any $(t_n)\in l_{p^*}$

$$\sup \left\| \sum_{k=1}^{k=n} t_k x_k \right\| \leqslant C \,\|(t_n)\|_{p^*}$$

3. For any $(t_n)\in l_{p^*}$, $\Sigma_n t_n x_n$ converges

4. There is a C such that for any finite subset Δ of $\mathbb{N}$ and any (θ_i) belonging to the unit ball of l_{p^*} we have $\left\| \sum_{n\in\Delta} \theta_i x_i \right\| \leqslant C$

In the existing literature weakly-p-summable sequences have appeared under various names: part 2. says that weakly-p-summable sequences are those admitting an upper-ℓ_p-estimate; part 3. says that they are the p-Hilbertian sequences.

Part 2 identifies the space $\ell_p^w(X)$ of weakly-p-summable sequences in X with the space $\mathfrak{L}(\ell_{p^*},X)$, which is a classical result due to Grothendieck.

The ideal $\Pi_{\infty,p}$ of absolutely (∞,p) summing operators was introduced in [C1], an it could be considered as a limit case of the classical (q,p)-summing operators, though our theory has nothing to see with the theory of (q,p) summing operators:

We say that an operator $T:X \longrightarrow Y$ is (∞,p) summing if it transforms weakly-p-summable sequences into norm null sequences.

They admit the following equivalent formulation:

Proposition. $\mathbf{Id(X) \in \Pi_{\infty,p}}$ **if and only if** $\mathfrak{L}(l_{p^*},X) = \mathfrak{K}(l_{p^*},X)$

For p=1, it can be proved that $\Pi_{\infty,1}=\mathfrak{U}$, the ideal of unconditionally converging operators. For $p\geq 1$, $\Pi_{\infty,p}$ forms an injective, non surjective and closed operator ideal.

Weakly-p-compact operators ($\mathfrak{W}_p$) were also introduced in [C1] as a gradation of the class of weakly compact operators. We say that an operator $T:X \longrightarrow Y$ is weakly-p-compact if from the image of each bounded sequence in X it is possible to extract a weakly-p-convergent subsequence.

For p>1, $\mathfrak{W}_p$ is an injective and surjective, not closed, operator ideal.

We do not know whether ideals $\Pi_{\infty,p}$ and $\mathfrak{W}_p$, $1<p<+\infty$, are idempotent, or even whether a Davis-Figiel-Johnson-Pelczynski factorization holds. $\mathfrak{W}_1$ and $\Pi_{\infty,1}$ are not idempotent.

It turns out that $\mathfrak{W}_p$ and $\Pi_{\infty,p}$ are, in a certain sense,"dual" notions: $\Pi_{\infty,p} \circ \mathfrak{W}_p = \mathfrak{K}$. Thus, the heart of the proof of the above proposition is:

Proposition 1. $\mathbf{Id(\ell_p) \in \mathfrak{W}_{p^*}}$, $1<p<+\infty$.

Here we present an outline of the general theory and some applications, mainly to operators acting on L_p. For example:

Proposition 2. $\mathbf{Id(X) \in \Pi_{\infty,2}}$ **if and only if** $\mathfrak{L}(L_p,X)=\mathfrak{K}(L_p,X)$ **for any (some** $p\geq 2$**).**
For $r>2$, $\mathbf{Id(X) \in \Pi_{\infty,r}}$ **if and only if** $\mathfrak{L}(L_{r^*},X) = \mathfrak{K}(L_{r^*},X)$

on the basis of which is:

Proposition 3. Let $1<p<+\infty$. $\mathbf{Id(L_p) \in \mathfrak{W}_{(\text{type } L_p)^*}}$; $\mathbf{Id(L_p) \in \Pi_{\infty,r<(\text{cotype } L_p)^*}}$

The connection with the type and cotype is not casual:

Proposition 4. Let X be a Banach space of cotype q. Then $\mathbf{Id(X) \in \Pi_{\infty,r}}$ **for all** $r<q^*$**. If X is reflexive, of type p, and has an unconditional basis, then** $\mathbf{Id(X) \in \mathfrak{W}_{p^*}}$**.**

Let us show some applications:

Proposition 5. Let $1<p<2$. Let X be a closed subspac of $L_p(\mu)$. Then X contains a copy of ℓ_p if and only if X^* contains a copy of ℓ_{p^*}.

and also:

Proposition 6. Let $1<p<+\infty$. For any Banach space X:

$$\Pi_{\infty,(\text{cotype } L_p)^*}(X,L_p) = \{\ell_{\text{cotype } L_p}\}\text{-strictly singular operators}$$

which complements L.Weis's results about characterizations of strictly singular operators in L_p-spaces.

"Subsequence principles" of the kind considered in [Ch] can be considered as a statement of the form $Id(H)\in\mathfrak{B}_2$, where H is any Hilbert space.

Since $Id(L_p)\in\mathfrak{B}_2$ when $p\geqslant 2$, the Grothendieck-Pietsch factorization theorem implies that p-summing operators are weakly-2-compact. So we have:

Proposition 7. Let X be a Banach space of finite cotype. Then

$$\mathfrak{B}(C(K),X)\subseteq\mathfrak{B}_2(C(K),X)$$

Applications to Dunford-Pettis properties shall take place in the next report.

References

[C1] **JMF.Castillo.** On weak-p-summability in vector valued sequence spaces (to appear)

[C2] **JMF.Castillo.** On Banach spaces X such that $L(L_p,X)=K(L_p,X)$.(to appear)

[Cj] **SD.Chatterji.** in "Probability and Analysis". Lecture Notes in Mathematics 1206. Springer. 1985

[D] **J.Diestel.** Sequences and Series in Banach spaces.GTM 92.Springer.1984.

[P] **A.Pietsch.** Operator Ideals. North Holland. 1980

[S] **F.Sanchez.** Tesis Doctoral. Universidad de Extremadura.

[Si] **I.Singer.** Bases in Banach spaces I. Springer 1970.

[W] **L.Weis.** On perturbations of Fredholm operators in $L_p(\mu)$-spaces. Proceedings of the AMS 67,2,(1977) 287-292