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A variant of Selberg's asymptotic formula *

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We consider the functions $\wedge^*_{r,k}(n)$, which generalize the well-known von Mangoldt function $\wedge(n)$, defined by $\wedge^*_{r,k}(n) = \sum_{d\delta=n} \mu^*_r(d) log^k \delta$ where $\mu^*_r(n)$ is the function of Moebius type such that $\mu^*_r(n) = 1$ if $n = 1$, $\mu^*_r(n) = 0$ if $p^{r+1}|n$ for some prime p and $\mu^*_r(n) = (-1)^{\Omega(n)}$ if $n = \prod p_i^{\alpha_i}$, $0 \le \alpha_i \le r$, $\Omega(n) = \sum \alpha_i$. From a variant of the T.Tatuzawa and K.Iseki formula [3] have obtained in [1] the following asymptotic formula for $\wedge^*_{r,k}(n)$

Theorem 1. *(Th.3 [1]) For fixed integers $r \ge 1$ and $k \ge 1$,*

$$(1)\ \sum_{n\le x} \Psi^*_{r,k}(x/n) log^k(x/n) h_r(n) + \sum_{i=1}^{k} \binom{k}{i} \sum_{n\le x} \Psi^*_{r,k}(x/n) log^{k-i}(x/n) \wedge^*_{r,i}(n) =$$

$$= k[\zeta(2)\gamma_r(r+1)]^2 \left(1 + \frac{k!}{(2k-1)!} \sum_{i=1}^{k} \frac{(2k-i-1)!}{(k-i)!}\right) x log^{2k-1} x + O(x log^{2k-2} x)$$

where $h_r(n) = \sum_{d\delta=n} \mu^*_r(d)$ and

$$\gamma_r(s) = \begin{cases} 1/\zeta(s), & \text{if } r \ge 1 \text{ odd} \\ \zeta(s)/\zeta(2s), & \text{if } r \ge 2 \text{ even} \end{cases}$$

In the particular case $k = 1$ we have

$$\sum_{n\le x} \Psi^*_{r,1}(x/n) log(x/n) h_r(n) + \sum_{n\le x} \Psi^*_{r,1}(x/n) \wedge^*_{r,1}(n) =$$

$$(2)\qquad = 2(\zeta(2)\gamma_r(r+1))^2 x log x + O(x)$$

For $k = 1$ and $r = 1$ Selberg's asymptotic formula is obtained. It's not difficult to prove that $\wedge^*_{r,k}(n)$ verifies the following lemmas .

Lemma 1. *Let α a positive integer and let p be a prime number . For k=1 and r odd integer , we have*

$$(3)\quad \wedge^*_{r,1}(p^\alpha) = f_1(\alpha, r) \log p \quad ; f_1(\alpha, r) = \begin{cases} (r+1)/2 & \text{if } \alpha \ge r \\ \alpha/2 & \text{if } \alpha < r \text{ and } \alpha \text{ even} \\ (\alpha+1)/2 & \text{if } \alpha < r \text{ and } \alpha \text{ odd} \end{cases}$$

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When k=1 and r is an even integer we get

$$(4)\quad \wedge^*_{r,1}(p^\alpha) = g_1(\alpha,r)\log p \quad ; g_1(\alpha,r) = \begin{cases} \alpha - r/2 & \text{if } \alpha \geq r \\ \alpha/2 & \text{if } \alpha < r \text{ and } \alpha \text{ even} \\ (\alpha+1)/2 & \text{if } \alpha < r \text{ and } \alpha \text{ odd} \end{cases}$$

If k=2 and r odd integer we get

$$(5)\quad \wedge^*_{r,2}(p^\alpha) = f_2(\alpha,r)\log^2 p \quad ; f_2(\alpha,r) = \begin{cases} (r+1)(\alpha - \frac{r}{2}) & \text{if } \alpha \geq r \\ \alpha(\alpha+1)/2 & \text{if } \alpha < r \end{cases}$$

For k=2 and r even integer we get

$$(6)\quad \wedge^*_{r,2}(p^\alpha) = g_2(\alpha,r)\log^2 p \quad ; g_2(\alpha,r) = \begin{cases} \alpha(\alpha-r) + r(r+1)/2 & \text{if } \alpha \geq r \\ \alpha(\alpha+1)/2 & \text{if } \alpha < r \end{cases}$$

For all r, k positive integers , let $\delta = \min\{\alpha, r\}$

$$(7)\quad \wedge^*_{r,k}(p^\alpha) = \log^k p \sum_{\beta=0}^{\delta} (-1)^\beta (\alpha - \beta)^k$$

This result (7) with r=1 is gived in [2- lemma 1] .

Lemma 2. *Let r be an odd integer and let* $n = \prod_{i=1}^t p_i^{\alpha_i}$ *be the representation as a product of prime factors of n .* $\wedge^*_{r,k}(n) = 0$ *whenever there are at least* $k+1$ *exponents* α_i *such that verify someone of the following conditions :*
i) $\alpha_i \geq r$ *ii)* α_i *odd* $< r$ *.*

Using these properties we get the following asymptotic formula equivalent to (2) for k=1 and r odd integer .

Theorem 2. *Let* $\mathfrak{C}_r$ *be the set of integers* $\mathfrak{m}$ *such that* $\mathfrak{m} = p^\alpha N$ *where p is a prime number ,* $\alpha = 0, 1$ *and N is a r-free square . When r is odd we have*

$$\sum_{\mathfrak{m} \leq x} \Psi^*_{r,1}(x/\mathfrak{m}) log(x/\mathfrak{m}) h_r(\mathfrak{m}) + \sum_{\mathfrak{m} \leq x} \Psi^*_{r,1}(x/\mathfrak{m}) \wedge^*_{r,1}(\mathfrak{m})$$

$$(8)\quad = 2(\zeta(2)\gamma_r(r+1))^2 x log x + O(x)$$

Proof.- For an integer n , let $n = \prod_{i=1}^t p_i^{\alpha_i}$ be his representation as a product of prime factors

$$\wedge^*_{r,1}(n) = \sum_{\beta_1=0}^{\delta_1} (-1)^{\beta_1} \ldots \sum_{\beta_t=0}^{\delta_t} (-1)^{\beta_t} [(\alpha_1 - \beta_1)\log p_1 + \ldots + (\alpha_t - \beta_t)\log p_t] =$$

$$(9)\quad = \sum_{i=1}^t (\chi(\delta_i)(\alpha_i - \delta_i) + \frac{\delta_i}{2} + \chi(\delta_i + 1)\frac{1}{2}) \prod_{\substack{j=1 \\ j \neq i}}^t \chi(\delta_i) \log p_i$$

where $\delta_i = \min\{\alpha_i, r\}$ $(i = 1, \ldots, t)$ and $\chi(\delta) = 1$ if δ even and $\chi(\delta) = 0$ if δ odd . Moreover , when r is odd , $h_r(n) = 1$ if n=1 or n is a r-free square and $h_r(n) = 0$ in the otherwise . Using the asymptotic formula (2) , we will get the formula (6) proving the following result

$$\sum = \sum_{\substack{n \le x \\ n \notin \mathfrak{C}_r}} \Psi^*_{r,1}(x/n) \wedge^*_{r,1}(n) = O(x) \tag{10}$$

For that , we will separate $\sum$ in two partes :

$$\sum_{\substack{p^\alpha \le x \\ \alpha \ge 2}} \Psi^*_{r,1}(x/p^\alpha) \wedge^*_{r,1}(p^\alpha) + \sum_{\substack{n \le x, n = p^\alpha m^2 \\ \alpha \ge 2, m^2 r-\text{free}}} \Psi^*_{r,1}(x/n) \wedge^*_{r,1}(n) = \sum_1 + \sum_2 \tag{9}$$

By the theorem 2 of [2] and the lemma 1 we have

$$\sum_1 \sim a_r x \sum_{2 \le \alpha \le \log x/\log 2} \sum_{p \le x^{1/\alpha}} f(\alpha, r) \frac{\log p}{p^\alpha} \ll$$

$$a_r x \sum_{p \le \sqrt{x}} \log p \sum_{2 \le \alpha \le \log x} \frac{|f(\alpha, r)|}{p^\alpha} \ll a_r \frac{r+1}{2} x \sum_{p \le \sqrt{x}} \frac{\log p}{p(p-1)} \ll_r x$$

Besides ,

$$\sum_2 \ll a_r \frac{r+1}{2} x \sum_{\substack{p^\alpha m^2 \le x \\ \alpha \ge 2, m^2 r-\text{free}}} \frac{\log p}{p^\alpha m^2} \ll a_r \frac{r+1}{2} x \sum_{\substack{p^\alpha \le x \\ \alpha \ge 2}} \frac{\log p}{p^\alpha} \sum_{m=1}^{\infty} \frac{1}{m^2} \ll$$

$$\ll a_r \frac{r+1}{2} x \sum_{2 \le \alpha \le [\log x/\log 2]} \sum_{p \le x^{1/\alpha}} \frac{\log p}{p^\alpha} \ll_r x$$

When we take r=1 in the theorem 2 , we get the Selberg's asymptotic formula restricted to the prime numbers .

References

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