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SOME EXTENSIONS ON THE SADOVSKII FUNCTOR [1]

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In the following X, Y, Z will be Banach spaces; B_X the closed unit ball of X; $\mathcal{L}(X,Y)$ the class of all (linear and continuous) operators between X and Y; $\ell_\infty(X)$ the space of all bounded sequences in X attached with the sup-norm $\|(x_n)\| := \sup\{\|x_n\| : n\in\mathbb{N}\}$. The range of the family (x_i) is denoted by $\{x_i\}$.

1 Denoting by m(X) the subspace of $\ell_\infty(X)$ of all sequences with relatively compact range, B.N. Sadovskii [SA] defined the functor P in the following way:

(α) $P(X) := \ell_\infty(X)/m(X)$;

(β) $T\in\mathcal{L}(X,Y) \longrightarrow P(T)\in\mathcal{L}(P(X),P(Y))$; $P(T)((x_n)+m(X)) := (Tx_n)+m(Y)$.

We call P the **Sadovskii functor**. Independently, it has been considered by [BHW]. We denote by h the Hausdorff measure of noncompactness [BG]. We have

PROPOSITION 1.(1) [HW] $\|(x_n)+m(X)\| = h(\{x_n\})$.

(2) $m(X) = \{(x_n)\in\ell_\infty(X) : h(\{x_n\})=0\}$.

(3) [SA] P(T) is one-one $\Leftrightarrow$ T is upper semi-Fredholm .

(4) [SA] P(T)=0 $\Leftrightarrow$ T is compact.

Lately, denoting by $m^W(X)$ the subspace of $\ell_\infty(X)$ of all sequences with relatively weakly compact range, J.J. Buoni and A. Klein [BK] defined the

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functor P^W in the following way:

(α) $P^W(X) := \ell_\infty(X)/m^W(X)$;

(β) $T\in\mathcal{L}(X,Y) \longrightarrow P^W(T)\in\mathcal{L}(P^W(X),P^W(Y))$; $P^W((x_n)+m^W(X)) := (Tx_n)+m^W(Y)$.

We call P^W the **Buoni-Klein functor**. We denote by w the De Blasi measure of weak noncompactness [DB]. We have

PROPOSITION 2.(1) [GM1] $\|(x_n)+m^W(X)\| = w(\{x_n\})$.

(2) $m^W(X) = \{(x_n)\in\ell_\infty(X) : w(\{x_n\})=0\}$.

(3) [GM1] $P^W(T)$ one–one $\Leftrightarrow$ T is tauberian .

(4) [BK] $P^W(T)=0 \Leftrightarrow$ T is weakly compact.

2 In the following we shall associate to every ideal of operators a functor and we shall study its properties. P and P^W will be included as the functors associated to the ideals of compact operators and weakly compact operators, respectively.

$\mathcal{A}$ will be an ideal of operators [PI]; $\mathcal{A}(X,Y):=\mathcal{A}\cap\mathcal{L}(X,Y)$. K. Astala has given the following definition [AS]: for $D\subset X$ bounded,

$$h_{\mathcal{A}}(D) := \inf \{\varepsilon>0 : \exists Z,\ \exists K\in\mathcal{A}(Z,X),\ D\subset KB_Z+\varepsilon B_X\} .$$

In [GM1] we consider the class

$$m_{\mathcal{A}}(X) := \{(x_n)\in\ell_\infty(X) : h_{\mathcal{A}}(\{x_n\})=0\}$$

and we prove that $m_{\mathcal{A}}(X)$ is a closed subspace of $\ell_\infty(X)$. The **generalized Sadovskii functor** associated to $\mathcal{A}$, $P_{\mathcal{A}}$, is defined in the following way:

(α) $P_{\mathcal{A}}(X) := \ell_\infty(X)/m_{\mathcal{A}}(X)$;

(β) $T\in\mathcal{L}(X,Y) \longrightarrow P_{\mathcal{A}}(T)\in\mathcal{L}(P_{\mathcal{A}}(X),P_{\mathcal{A}}(Y))$; $P_{\mathcal{A}}(T)((x_n)+m_{\mathcal{A}}(X)) := (Tx_n)+m_{\mathcal{A}}(Y)$

PROPOSITION 3. [GM1] $h_{\mathcal{A}}(\{x_n\}) = \|(x_n)+m_{\mathcal{A}}(X)\|$.

If $\mathcal{A}$=Co, the compact operators, we obtain that $P_{\mathcal{A}}$ is the Sadovskii functor, $m_{\mathcal{A}}(X)=m(X)$ and $h_{\mathcal{A}}=h$. If $\mathcal{A}$=WCo, the weakly compact operators, then $P_{\mathcal{A}}$ is the Buoni-Klein functor, $m_{\mathcal{A}}(X)=m^W(X)$ and $h_{\mathcal{A}}=w$.

3 The above results have been analized in [GM2] in a broader context: Let (E,d) be a complete metric space. We consider the set of all bounded families (x_i) in E with index set I,

$$\ell_\infty(I,E) := \{(x_i)\subset E : (x_i) \text{ bounded}\},$$

attached with the distance $d((x_i),(y_i)) := \sup \{d(x_i,y_i) : i\in I\}$. Then $\ell_\infty(I,E)$ is a complete metric space.

If $P_b(E) := \{A\neq\emptyset : A\subset E \text{ bounded}\}$, then a map μ: $P_b(E) \longrightarrow \mathbb{R}$ is called a **set measure** [MA] if $\mu\geq 0$, μ increasing and

(1) $\exists N\in P_b(E),\ \mu(N)=0$.

(2) $\exists r(\mu)\geq 0,\ \forall A\in P_b(E),\ \forall\varepsilon>0,\ \mu(K(A,\varepsilon)) \leq \mu(A) + r(\mu)\varepsilon$.

(3) $\forall A,B\in P_b(E),\ \mu(A\cup B) \leq \max \{\mu(A),\mu(B)\}$.

Moreover, the kernel of μ is defined by $\text{Ker}(\mu) := \{N\in P_b(E) : \mu(N)=0\}$. In [MA] it is show that for every set measure μ there exists a **canonical measure** μ_c such that $\text{Ker}(\mu)=\text{Ker}(\mu_c)$ and we can take $r(\mu_c)=1$.

If μ is a canonical measure in E, and we denote by $\mu(I,E)$ the subset of $\ell_\infty(I,E)$ of all the families whose range belongs to $\text{Ker}(\mu)$, then we have

PROPOSITION 4.(1) [GM2] $\mu(\{x_i\}) = d((x_i),\mu(I,E))$

(2) [GM2] $\mu(I,E)$ is closed in $\ell_\infty(I,E)$.

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