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AN OPERATOR IDEAL ASSOCIATED TO TAUBERIAN OPERATORS[1]

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R. Herman [8] introduced the class of <u>almost weakly compact</u> (a.w.c.) operators as those operators which have bounded inverses only in reflexive subspaces. This class extends the strictly singular and the weakly compact operators, but it is not known if it forms an operator ideal (if the sum of two a.w.c. operators is an a.w.c. operator). Following an analogous approach, J. Howard [9] defined for a non-empty subclass G of the class $\mathbb{B}$ of all Banach spaces the classes S(G) and C(G) of G-singular and G-cosingular operators, which generalize the operator ideals SS of strictly singular operators and SC of strictly cosingular operators (case G = infinite dimensional spaces). Moreover, for G = nonreflexive spaces, S(G) is the class of a.w.c. operators. In general S(G) and C(G) are not operator ideals, but L. Weis [15] found sufficient conditions on G guaranteeing it. These classes were studied also in [1, 3, 11, 13].

In this paper, denoting by $\mathbb{R}$ the class of all reflexive Banach spaces, we introduce the class $\mathbb{R}$-SS, the <u>$\mathbb{R}$-strictly singular operators</u>, which generalize the strictly singular operators with a definition slightly different to that of a.w.c. operators. Then using some recent results about tauberian operators [7] we prove that $\mathbb{R}$-SS is an operator ideal; hence it is the right extension of the strictly singular operators from the finite dimensional to the reflexive case. We give also an analogous extension of the strictly cosingular operators, and other extensions of SS and SC for other classes of Banach spaces not covered by Weis conditions.

We shall denote SF_+ (SF_-) the class of upper (lower) semi-Fredholm operators.

Let $T^* \in L(Y^*,X^*)$ and $J(X)$ denote the conjugate operator of $T \in L(X,Y)$ and the canonical image of X in the second dual X^{**}.

T is <u>tauberian</u> when $T^{**^{-1}}J(Y) = J(X)$.

T is <u>cotauberian</u> when T^* is tauberian.

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Definition 1 $K \in L(X,Y)$ is $\mathbb{R}$-strictly singular ($K \in \mathbb{R}$-SS) when given a space Z and $A \in L(Z,X)$, if KA is tauberian then A is weakly compact.

K is $\mathbb{R}$-strictly cosingular ($K \in \mathbb{R}$-SC) when given a space Z and $A \in L(Y,Z)$, if AK is cotauberian then A is weakly compact.

Theorem 2 $\mathbb{R}$-SS and $\mathbb{R}$-SC are operator ideals.

Proposition 3 (a) Weakly compact operators belong to $\mathbb{R}$-SS $\cap$ $\mathbb{R}$-SC.

(b) Every $K \in \mathbb{R}$-SS(X,Y) is an a.w.c. operator; i.e., if K is an isomorhism in a subspace $M \subset X$, then M is reflexive.

(c) If $K \in \mathbb{R}$-SC(X,Y) and N is a subspace of Y such that the product with the quotient map $q_N K$ is surjective, then Y/N is reflexive.

(d) The identity I_X of a Banach space X belongs to $\mathbb{R}$-SS ($\mathbb{R}$-SC) if and only if X is reflexive.

Observation 4 We do not know if $\mathbb{R}$-SS coincides with the class of a.w.c. operators. Analagously we do not now if the property of $\mathbb{R}$-SC in Proposition 3(c) characterizes the class.

Example 5 Any quotient map $q : \ell_1 \longrightarrow c_o$ is an example of operator in $\mathbb{R}$-SS which is not weakly compact.

OTHER EXTENSIONS OF SS AND SC

Let U be one of the operator ideals Co, WCo, Ro, CC, WCC, Gr, and Cd of all compact, weakly compact, Rosenthal, completely continuous, weakly completely continuous, Grothendieck and condensed operators respectively. In [5] and [6] we defined two semigroups SU_+ and SU_- of operators such that SCo_+ and SCo_- coincide with the classes SF_+ and SF_- of upper and lower semi-Fredholm operators and they are contained in SU_+ and SU_- respectively for every operator ideal U. Also $SWCo_+$ coincides with the class of all tauberian operators. Moreover in [7] we obtained perturbative characterizations of the classes SU_+.

For an operator ideal U, the space ideal Sp(U) is the class of all Banach spaces X whose identity I_X belongs to U, the dual operator ideal U^d of U is the class of operators K such that K^* is in U , and $Sp(U^d) = \{ X \,/\, X^* \in Sp(U) \}$.

Using the semigroups SU_+ and SU_- and its perturbative characterizations we can define extensions of SS and SC from the case of finite dimensional spaces $\mathbb{F}$ to the case of the corresponding space ideal.

<u>Definition</u> 6 Let $U \in \{Ro, CC, WCC\}$ and $\mathbb{A} = Sp(U)$.

$\mathbb{A}$-SS := { $K \in L(X,Y)$: For any Z and $A \in L(Z,X)$, $KA \in SU_+ \Rightarrow A \in U$ }

$\mathbb{A}^d$-SC := { $K \in L(X,Y)$: For any Z and $B \in L(Y,Z)$, $BK \in SU_- \Rightarrow A \in U^d$ }

For U : Gr or Cd we define $\mathbb{A}$-SC putting U instead of U^d in the definition of $\mathbb{A}^d$-SC.

<u>Theorem</u> 7 $\mathbb{A}$-SS and $\mathbb{A}$-SC are operator ideals containing U and U^d.

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