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VON NEUMANN REGULARITY IN JORDAN BANACH TRIPLE SYSTEMS

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Kaplansky proved in [6] that von Neumann regular associative Banach algebras are finite-dimensional. Recently, Benslimane and Kaidi showed in [1] that every von Neumann regular noncommutative Jordan Banach algebra is a direct sum of a finite number of closed simple ideals each of which is either finite-dimensional or an infinite-dimensional flexible quadratic complex algebra. Von Neumann regularity in Jordan triple systems was first considered by Meyberg [7] who proved that nondegenerate Jordan triple systems satisfying descending chain conditions on principal inner ideals are von Neumann regular.

The purpose of this note is to present our main results on von Neumann regular Jordan Banach triple systems settled in [4].

We remind the reader the basic definitions and notation on triples systems; as references we mention the book [7]. Let Φ be a commutative associative ring with 1 ($1/2 \in \Phi$). A *Jordan triple system* (JTS) is a unital Φ-module J with a quadratic map $P : J \to \mathrm{End}_\Phi(J)$ such that the following identities hold in all scalar extensions.

(1.1) $V(x,y)P(x) = P(x)V(y,x)$

(1.2) $V(P(x)y, y) = V(x, P(y)x)$

(1.3) $P(P(x)y) = P(x)P(y)P(x)$

where $V(x, y)z = \{x\, y\, z\} = P(x, z)y = P(x+z)y - P(x)y - P(z)y$.

A unital Jordan algebra is a JTS with unit element 1 ($P(1) = \mathrm{id}$). Every Jordan algebra J defined by a quadratic map $U : J \to \mathrm{End}_\Phi(J)$ and a squaring operation gives rise to a JTS by defining $P(a) = U(a)$. Other examples of JTS are provided by rectangular $n \times m$ matrices over Φ under $P(a)b = ab^t a$, where b^t denotes the transpose of the matrix b.

Let J be a JTS. A submodule I of J is called an *inner ideal* if $P(I)J \subset I$. An *ideal* is a submodule N of J such that $\{J\,J\,N\} + \{J\,N\,J\} \subset N$. A JTS J is called *simple* if $\{J\,J\,J\} \neq 0$ and J does not contain proper ideals.

A socle theory can be developed for JTS that extends that for associative and Jordan algebras (see [2]). The main result of this theory, socle theorem, was suggested to the first author by McCrimmon and proved in [3] for the special case of an associative triple system of second kind. A JTS is called *nondegenerate* if $P(a) = 0 \Rightarrow a = 0$. It is well known that an associative algebra A is semiprime if and only if the JTS A^+ defined by $P(a)b = aba$ is nondegenerate.

THEOREM 1 (Socle theorem). Let J be a nondegenerate Jordan triple system. The socle $\mathrm{Soc}(J)$ of J, defined to be the sum of all minimal inner ideals, is an ideal of J. If J contains minimal inner ideals then $\mathrm{Soc}(J)$ is a direct sum of simple ideals each of which contains a minimal inner ideal; $\mathrm{Soc}(J)$ is simple when J is prime.

An element $a \in J$ is called *von Neumann regular* (*regular*) if there exists $b \in J$ such that $P(a)b = a$. An ideal N of J is called *regular* if all its elements are regular. The following result extends to JTS a result of [5]. A *normed Jordan triple system* is a JTS over the complex field with a norm $\|\ .\ \|$ making continuous the triple product $\{a\, b\, c\}$. If the norm $\|\ .\ \|$ is complete we say that J is a *Jordan Banach triple system.*

THEOREM 2. (i) The socle of a nondegenerate Jordan triple system J (over a field of characteristic $\neq 2,3$) is regular.

(ii) If J is a nondegenerate Jordan Banach triple system, then the socle of J coincides with the unique maximal von Neumann regular ideal of J.

Let J be a JTS over a field F. An element $c \in J$ is called *reduced* if $P(c)J = Fc$. The linear span $\mathrm{Red}(J)$ of all reduced elements is an ideal of J. A JTS J is called *reduced* if $J=\mathrm{Red}(J)$.

THEOREM 3. Every simple Jordan triple system J which is reduced over an algebraically closed field F (of characteristic $\neq 2,3$) is isomorphic to one of the following:

(i) The JTS $\mathfrak{F}(X)$ of all finite rank linear operators $a : X \to X$ having a (unique) adjoint with respect to a pair of dual vector spaces (X,X',g) over F, under $P(a)b = aba$.

(ii) $\mathfrak{S}(\mathfrak{F}(X,g), \#)$ or $\mathfrak{A}(\mathfrak{F}(X,g), \#)$ under $P(a)b = aba$, with respect to a self-dual (symmetric or alternate) vector space (X,g) over F.

(iii) $\mathfrak{S}(\mathfrak{F}(X,Y),\#)\oplus\mathfrak{S}(\mathfrak{F}(Y,X),\#)$ or $\mathfrak{A}(\mathfrak{F}(X,Y),\#)\oplus\mathfrak{A}(\mathfrak{F}(Y,X),\#)$ under $P(a_1,b_1)(a_2,b_2)= (a_1b_2a_1, b_1a_2b_1)$ with respect to a pair of dual vector spaces (X,Y,g) and its opposite (Y,X,g^{op}) over F.

(iv) $\mathfrak{F}((X,g),(Y,h))$ under $P(a)b = ab^{\#}a$, where $(X,g),(Y,h)$ are self-dual (both symmetric or alternate) vector spaces over F.

(v) $\mathfrak{F}(X,Y) \oplus \mathfrak{F}(X',Y')$ under $P(a_1, b_1)(a_2, b_2) = (a_1b_2{}^{\#}a_1, b_1a_2{}^{\#}b_1)$ with respect to (X,X',g), (Y,Y',h) pairs of dual vector spaces over F.

(vi) A simple JTS $J(Q,\eta)$ under $P(a)b = Q(a, \eta(b))a - Q(a)\eta(b)$, where J is a vector space over F, Q is a nondegenerate quadratic form and η a linear mapping such that $\eta^2 = \mathrm{Id}$, $Q(\eta(a)) = Q(a)$.

(vii) The JTS $J(Q) \oplus J(Q)$ under $P(a,b)(c,d) = (P(a)d, P(b)c)$ where $P(a)$ is as above with $\eta = \mathrm{Id}$.

(viii) A finite-dimensional simple exceptional JTS over F.

THEOREM 4. For a Jordan Banach triple system $(J, \|\ .\ \|)$ the following conditions are equivalent:

(a) J is von Neumann regular.

(b) J is nondegenerate and coincides with its socle.

(c) $J = M_1 \oplus M_2 \oplus \ldots \oplus M_n$ is a direct sum of closed simple ideals each of which is topologically isomorphic to one of the following:

(i) A simple finite dimensional Jordan triple system over the complex field.

(ii) The Jordan Banach triple system $\mathbf{BL}(X,Y)$ of all continuous linear operators $a : X \to Y$ under $P(a)b = ab^{\#}a$ endowed with the norm $\| a \|' = \mathrm{Max}\{ |a| , | a^{\#} |\}$ ($|.|$ being the operator norm), where (X,g), (Y,h) are complex self-dual Banach spaces (both symmetric or alternate) with X finite-dimensional.

(iii) The Jordan Banach triple system $\mathbf{BL}(X,Y)\oplus\mathbf{BL}(X',Y')$ under $P(a_1,b_1)(a_2,b_2)= (a_1b_2{}^{\#}a_1 , b_1a_2{}^{\#}b_1)$ with $\|(a, b)\|'=\mathrm{Max}\{ |a|, |b|, |a^*|, |b^*|\}$, where (X,X',g), (Y,Y',h) are pairs of dual complex Banach spaces with X, and therefore X' too, finite-dimensional.

(iv) The Jordan Banach triple system $J(Q,\eta)$ under $P(a)b= Q(a,\eta(b))a-Q(a)\,\eta(b)$, where J is a infinite-dimensional complex Banach space, Q a continuous nondegenerate quadratic form and $\eta : J \to J$ is a continuous linear mapping such that $\eta^2 = \mathrm{Id}$ and $Q(\eta(a))=Q(a)$.

(v) The polarized Jordan Banach triple system $J(Q)\oplus J(Q)$, where $J(Q)$ is an infinite-dimensional quadratic Jordan Banach algebra (see [1]), and $\|(a,b)\|= \mathrm{Max}\{ \|a\|, \|b\|\}$.

Sketch of the proof. From Theorem 2, every von Neumann regular Jordan Banach triple system coincides with its socle and hence, by completeness, is a direct sum of a finite number of closed simple ideals. Since every simple von Neumann regular Jordan Banach triple system is reduced, we apply Theorem 3. Finally, we improve this classification for the Banach case with arguments suggested to us by Rodriguez Palacios.

REFERENCES

1. M. BENSLIMANE and A.M. KAIDI, Structure des algèbres de Jordan-Banach non commutatives complexes régulières ou semi-simples à Spectre fini, *J. Algebra* (1) 113 (1988), 201-206.

2. A. FERNANDEZ LOPEZ, Modular annihilator Jordan algebras, Commun. in Algebra (12) 13 (1985), 2597-2613.

3. A. FERNANDEZ LOPEZ and E. GARCIA RUS, Prime associative triple systems with nonzero socle. Submitted to *Commun. in Algebra*.

4. A. FERNANDEZ LOPEZ, E. GARCIA RUS and E. SANCHEZ CAMPOS, Von Neumann regular Jordan Banach triple systems. Preprint 1989.

5. A. FERNANDEZ LOPEZ and A. RODRIGUEZ PALACIOS, On the socle of a noncommutative Jordan algebra, *Manuscripta Math.* 56 (1986), 269-278.

6. I. KAPLANSKY, Regular Banach algebras, *J. Indian Math. Soc.* 12 (1948), p.57-62.

7. K. MEYBERG, *Lectures on algebras and triple systems*, Lecture Notes , The University of Virginia, Charlottesville 1972.